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LETTER TO THE EDITOR

Kepler–Ermakov problems

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Abstract. A class of dynamical systems is presented which includes, as special cases, both the (autonomous) Ermakov system and central force problems of Kepler type with angular dependence of the force. It is shown that all members of this class are linearizable up to a pair of quadratures.

The solutions of the classical, two-dimensional Kepler problem are conic sections. This follows from the fact that, using the angular momentum, which is an integral of motion, one may make a simple change of variables which reduces the radial equation of motion to an inhomogeneous, linear differential equation which has no explicit dependence on the independent variable [12]. If one allows non-isotropy of the central force then the situation is only slightly more complex: the inhomogeneity acquires a dependence upon the independent variable.

Ermakov systems [8], on the other hand, are time-dependent dynamical systems of the same order as the Kepler problem but having a non-central force law. They contain one arbitrary function of time and two arbitrary homogeneous functions of the particle coordinates. Such systems are also linearizable [2]. In particular they possess an invariant which plays a role analogous to that of the angular momentum in the Kepler problem. An example of such a system is afforded by the motion of a massless charge in the field of a fixed electric dipole [5]. In general, though, Ermakov systems are not Hamiltonian.

Just as the numerical value of the angular momentum in the Kepler problem, determined by initial conditions, enters into the linearization as a parameter, so does the invariant in the case of the Ermakov system. In each case the linearization is thus a one-parameter family of linear equations. Whilst the Kepler problem is an integrable Hamiltonian system and this dependence upon initial conditions well understood [1], the Ermakov systems are not generally integrable, even when Hamiltonian. The use of the linearization in discussing the global properties of the solutions to the nonlinear system is consequently *ad hoc* [3, 4].

The purpose of the present letter is to give a class of systems which can be regarded either as perturbations of the classical Kepler problem or of an autonomous Ermakov system, which preserve the property of linearizability. Previous work on generalized Ermakov systems has concentrated on the introduction of greater complexity into the dependence of the arbitrary functions [10, 11] or on the extension to larger numbers of dependent variables [6, 9] of the time-dependent systems. In the latter case the systems are clearly reduced to autonomous form up to the solution of a linear equation [2] so that the time dependence is, in a sense, spurious. The systems to be discussed here are autonomous. One could reverse the usual autonomizing process to render them non-autonomous but this would be artificial in the present context.

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The transformations to be employed here owe something to the classical theory, as a perusal of chapter four of [12] will show.

We will call the following the Kepler–Ermakov system:

$$\begin{aligned}\ddot{x} &= -\frac{x}{r^3}H + \frac{1}{x^3}X \\ \ddot{y} &= -\frac{y}{r^3}H + \frac{1}{y^3}Y.\end{aligned} \tag{1}$$

Here H, X and Y are homogeneous functions of x and y of degree zero which remain unspecified. In the case that H is taken to be identically zero we have the class of (autonomous) Ermakov systems. If, on the other hand, X and Y are identically zero, we have a Kepler problem with an angularly dependent central force. The Kepler–Ermakov system has inverse square and inverse cube terms to the force law. An example would be a massive, charged particle moving in the field of a massive, fixed dipole. It is a central force problem only when the homogeneous functions X and Y satisfy $x^4Y = y^4X$.

In polar coordinates the Kepler–Ermakov system takes the form

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= -\frac{1}{r^2}H + \frac{1}{r^3}R(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{1}{r^3}\Theta(\theta)\end{aligned} \tag{2}$$

where R and Θ are suitably related to X and Y.

Since (1) is not a central force problem the angular momentum is not an integral of motion. Such an integral is, however, afforded by the expression

$$I = \tfrac{1}{2}(\dot{x}y - x\dot{y})^2 + \int^{\eta} \{zY(z) - z^{-3}X(z)\}\,\mathrm{d}z \tag{3}$$

where $\eta = x/y$ and X and Y are written as functions of η. I is called the Ray–Reid–Lewis invariant [8] in the case where (1) is a Ermakov system. In general (1) has no further invariant which is a function of x, y, $\dot{x}$ and $\dot{y}$ only. We may solve (3) to obtain $\dot{x}y - x\dot{y} = h(\eta; I)$. An integral of the type (3) exists for a rather wider class of perturbation to the Ermakov system. Only the Kepler-type however admits the linearization below.

Now introduce a new independent variable $\psi = 1/y$. In fact η and ψ are the choice of dependent variables used by Whittaker [12] in showing that a general problem of central force type can be transformed to one with a parallel field of force. However we will take ψ as *dependent* and η as *independent* variable, as is effectively done in linearizing the Kepler problem. Under the change of independent variable, η and t will be related by the separable first order ordinary differential equation

$$\dot{\eta} = \psi^2(\eta; I)h(\eta; I). \tag{4}$$

Now by expressing all time derivatives as η derivatives in, say, the second of equations (1). We obtain for $\psi(\eta; I)$ the inhomogeneous linear second-order equation

$$h\frac{\mathrm{d}}{\mathrm{d}\eta}\left(h\frac{\mathrm{d}\psi}{\mathrm{d}\eta}\right) + Y(\eta)\psi = \tilde{H}(\eta) \tag{5}$$

where $\tilde{H}(\eta) = (1+\eta^2)^{-3/2} H(\eta)$. Because h depends on I, this equation is really a one-parameter family. If we have a solution $\psi(\eta; I)$ of (5) for a specific value of I then we may obtain η as a function of t from (4), thence $\psi(\eta(t); I)$ and so $y(t)$ and $x(t)$ for this value of I. The general solution of the family (5) will depend upon I and two constants of integration. Integration of (4) then gives us the requisite four constants.

When (5) is homogeneous (the Ermakov system) one may achieve the integration of (4) knowing only the general solution of (5). Thus if ψ is a solution of (5) and ψ^* a linearly independent solution we may certainly scale ψ^* so that it satisfies the Wronskian identity

$$\frac{d\psi}{d\eta}\psi^* - \psi\frac{d\psi^*}{d\eta} = \frac{1}{h}. \tag{6}$$

But then the integral of (4) is

$$(t-c)\psi(\eta; I) = \psi^*(\eta; I) \tag{7}$$

which implicity relates η and t. In the inhomogeneous case no pair of solutions satisfies (6) and so the system is linearizable up to *two* quadratures, the expression for $h(\eta; I)$ and the solution of (4).

In the case that (1) is a central force problem with angular dependence, $h = \sqrt{2I}$ is independent of η.

In solving (4) and (5) one must bear in mind that η is a projective variable. If the orbit of (1) passes across or meets the x-axis it is necessary to transform to the variables $\zeta = y/x$ and $\varphi = 1/x$. Away from either axis the variables are related by $\eta\zeta = 1$ and $\zeta\psi = \varphi$.

Since the role of equation (4) is solely to relate the parameter of the projective line to the time we may deduce geometrical information about the orbits from equation (5) alone. Thus for a given value of I and a given solution $\psi(\eta; I)$ of (5) the equations

$$y(\eta) = \frac{1}{\psi(\eta; I)} \qquad x(\eta) = \frac{\eta}{\psi(\eta; I)} \tag{8}$$

give a parametrization by η of the orbit $y\psi(x/y; I) = 1$. An important consequence of the linearization (5) is that we may obtain the general orbit with a given value of I from three particular integrals of equations (1) having that same value of I.

Consider firstly the Ermakov system, $H(\eta) \equiv 0$. For a given value of I let $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ be a pair of orbits not related by a simple dilation. Provided they do not pass through the origin we may parametrize them in the form (8) for suitable functions ψ_1 and ψ_2. Since the orbits are not related by dilation, ψ_1 and ψ_2 are linearly independent solutions to (5) whose general solution is then $\psi = c_1\psi_1 + c_2\psi_2$, c_1 and c_2 being arbitrary constants. The orbit corresponding to this ψ is

$$y(\eta) = \frac{y_1(\eta)y_2(\eta)}{c_1 y_2(\eta) + c_2 y_1(\eta)} \qquad x(\eta) = \frac{x_1(\eta)x_2(\eta)}{c_1 x_2(\eta) + c_2 x_1(\eta)} \tag{9}$$

where $(x_i(\eta), y_i(\eta))$ $(i = 1, 2)$ are the η-parametrizations of the given orbits $\Phi_1 = 0$ and $\Phi_2 = 0$. Equations (9) and the corresponding ones involving ζ describe the general orbit with the appropriate value of I.

Now consider the Kepler–Ermakov system. For a given value of I let $\Phi_1(x, y) = 0$, $\Phi_2(x, y) = 0$ and $\Phi_3(x, y) = 0$ be three orbits such that the parametrizations $(x_i(\eta), y_i(\eta))$ do *not* satisfy a relation of the form

$$\lambda_1 y_2 y_3 + \lambda_2 y_3 y_1 + \lambda_3 y_1 y_2 = 0 \tag{10}$$

for all η and any constants λ_1, λ_2 and λ_3 whose sum vanishes. We have corresponding functions $\psi_i(\eta, I)$ $(i=1,2,3)$ which satisfy the *inhomogeneous* equation (5). Then the differences $\tilde{\psi}_3 = \psi_1 - \psi_2$ and $\tilde{\psi}_1 = \psi_2 - \psi_3$ satisfy the homogeneous form of (5) and are linearly independent by virtue of not satisfying (10). Now the solution of the inhomogeneous equation (5) is found by the method of variation of constants to be

$$\psi = \frac{1}{w}\tilde{\psi}_1 \int_a^\eta \frac{\tilde{H}}{h}\tilde{\psi}_3\,\mathrm{d}\eta - \frac{1}{w}\tilde{\psi}_3 \int_b^\eta \frac{\tilde{H}}{h}\tilde{\psi}_1\,\mathrm{d}\eta \tag{11}$$

where $w = h[\tilde{\psi}_2'\tilde{\psi}_1 - \tilde{\psi}_2\tilde{\psi}_1']$ is a constant determined by the ψ_i $(i=1,2,3)$ and a and b are arbitrary constants within the domains of h, $\tilde{\psi}_1$ and $\tilde{\psi}_3$. The corresponding equation for y can be written symmetrically as

$$\frac{1}{y} + \frac{1}{y_1}\int_{\eta_1}^{\eta} \frac{\tilde{H}}{h}\left(\frac{1}{y_2} - \frac{1}{y_3}\right) + \frac{1}{y_2}\int_{\eta_2}^{\eta} \frac{\tilde{H}}{h}\left(\frac{1}{y_3} - \frac{1}{y_1}\right) + \frac{1}{y_3}\int_{\eta_3}^{\eta} \frac{\tilde{H}}{h}\left(\frac{1}{y_1} - \frac{1}{y_2}\right) = 0 \tag{12}$$

η_1, η_2 and η_3 being constants satisfying the single relation

$$\int_{\eta_2}^{\eta_3} \frac{\tilde{H}}{h}\frac{1}{y_1} + \int_{\eta_3}^{\eta_1} \frac{\tilde{H}}{h}\frac{1}{y_2} + \int_{\eta_1}^{\eta_2} \frac{\tilde{H}}{h}\frac{1}{y_3} = 0. \tag{13}$$

The equation for $x(\eta)$ is obvious.

It also follows, of course, from the above argument that the general solution of a Kepler–Ermakov system (1) is deducible from the general solution of the Ermakov system obtained by omitting the inverse square terms in the force law. Most trivially it is easy to construct the general solution to the Kepler problem from a general pair of solutions, namely straight lines, of the system $\ddot{x} = \ddot{y} = 0$ i.e. to construct the general conic section from the family of degenerate conic sections.

A less trivial example is the following. Consider the system

$$\begin{aligned} \ddot{x} &= -\frac{x}{r^3} + \frac{1}{x^3} - (1-\mu^2)\frac{x}{y^4} \\ \ddot{y} &= -\frac{y}{r^3} + \frac{\mu^2}{y^3}. \end{aligned} \tag{14}$$

Let us define a new variable Ω by $\eta^2 = I + \alpha\cos\Omega$ where $\alpha^2 = I^2 - 1$. The linearization is then

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\Omega^2} + \mu^2\psi = \frac{1}{(1+I+\alpha\cos\Omega)^{3/2}} \tag{15}$$

in which the homogeneous part happens to be independent of I. We may take $\tilde{\psi}_1 = \sin\mu\Omega$, $\tilde{\psi}_2 = \cos\mu\Omega$ in (11) noting that $\mathrm{d}\eta = h(\eta; I)\,\mathrm{d}\Omega$ to obtain parametrizations via Ω of the general solution curve.

References

[1] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[2] Athorne C, Rogers C, Ramgulam U and Osbaldestin A 1990 *Phys. Lett.* **143A** 207
[3] Athorne C 1991 *J. Phys. A: Math. Gen.* **24** 945
[4] Athorne C Stability and periodicity in coupled Pinney equations *J. Diff. Eq.* in press
[5] Jackson J D 1962 *Classical Electrodynamics* (New York: Wiley)

[6] Leach P G L 1991 Generalized Ermakov systems *Preprint* Department of Mathematics, University of Natal
[7] Lutzky M 1980 *Phys. Lett.* **78A** 301
[8] Ray J R and Reid J L 1979 *Phys. Lett.* **71A** 317
[9] Ray J R and Reid J L 1979 *Phys. Lett.* **74A** 23
[10] Reid J L and Ray J R 1980 *J. Math. Phys.* **21** 1583
[11] Sarlet W 1981 *Phys. Lett.* **82A** 161
[12] Whittaker E T 1904 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)